JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 28, No. 4, July–August 2005

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Cross-Track Motion of Satellite Formations in the Presence of J_2 Disturbances

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Introduction

THIS Note augments and completes a previous work[1] by the authors in which a set of constant coefficient linear differential equations were developed that describe the relative motion between two satellites in the presence of the Earth's J_2 geopotential. In the previous work, the motion was separated into in-plane and cross-track motion. The analysis of the in-plane motion was relatively straightforward and remains untouched in this Note. The cross-track motion was more difficult to derive; linearizations with respect to time were used to develop analytical solutions. In this Note, the equations are instead simplified using small angle approximations. These simplifications are valid due to the very small angles that describe the cross-track motion. Although no longer linear, the equations can still be solved analytically. The result is a set of differential equations that are more accurate and more intuitive.

Describing the Cross-Track Motion

Motion in the cross-track direction can be modeled as a periodic function with varying amplitude $A(t)$, frequency $B(t)$, and phase $C(t)$:

$$z = A(t)\sin[B(t)t - C(t)] \tag{1}$$

This periodicity is due to the differences in orbital inclination and longitude of the ascending node between the satellite and reference orbit. As shown in Fig. 1, $A(t)$, $B(t)$, and $C(t)$ can be defined as

$$A(t) = r_{\text{ref}}\Phi, \qquad B(t) = \text{orbital frequency}, \qquad C(t) = \gamma \tag{2}$$

Amplitude and Phasing Angle

The phasing angle and amplitude, γ and Φ, can be calculated by utilizing spherical trigonometry:

$$\gamma = \cot^{-1}\left[\frac{\cos i_{\text{ref}}\cos\Delta\Omega - \cot i_{\text{sat}}\sin i_{\text{ref}}}{\sin\Delta\Omega}\right]$$

$$\Phi = \cos^{-1}(\cos i_{\text{sat}}\cos i_{\text{ref}} + \sin i_{\text{sat}}\sin i_{\text{ref}}\cos\Delta\Omega) \tag{3}$$

In the presence of the J_2 disturbance, the orbital planes rotate about the $\hat{Z}$ axis (the north pole). Each satellite's orbit rotates about this axis at a slightly different rate based on its inclination. This is manifested in the mean variation of the orbital elements by a linear variation in time of the longitude of the ascending node. The differential change in the ascending nodes is given by

$$\Delta\Omega = \Delta\Omega_0 + \frac{\mathrm{d}\Delta\Omega}{\mathrm{d}t}t, \qquad \frac{\mathrm{d}\Delta\Omega}{\mathrm{d}t} = -jn(\cos i_{\text{sat}} - \cos i_{\text{ref}})$$

$$j = \frac{3J_2R_e^2}{2r_{\text{ref}}^2}, \qquad n = \sqrt{\frac{\mu}{r_{\text{ref}}^3}} \tag{4}$$

The J_2 disturbance does not affect the mean variation of the inclination, and i_{ref} and i_{sat} remain constant.

Orbital Frequency

The orbital frequency (as measured from one equatorial crossing to the next equatorial crossing) is affected by the J_2 disturbance in two ways: The mean angular velocity is changed, and the distance a satellite travels over one period is shortened. The mean angular velocity is calculated by examination of the time-averaged effect of the J_2 disturbance. The result is the circular orbit angular velocity n, multiplied by a dimensionless coefficient c, which gives the new angular velocity:

$$\omega = nc, \qquad c = \sqrt{1+s}, \qquad s = j[(1 + 3\cos 2i_{\text{ref}})/4] \tag{5}$$

The change in distance that a satellite must travel due to variations in the longitude of the ascending node is shown in Fig. 2. Because of the J_2 disturbance force, a satellite that starts at point A will arrive at point B one orbital period later. The new orbital frequency is, thus,

$$B(t) = nk, \qquad k = c + j\cos^2 i_{\text{ref}} \tag{6}$$

This k is the similar to k in the previous paper,[1] except that n has been removed from the constant.

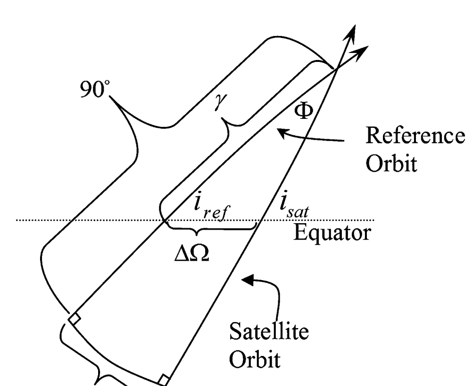

Fig. 1 Amplitude and phase of cross-track motion.

Received 2 August 2004; revision received 17 November 2004; accepted for publication 22 November 2004.

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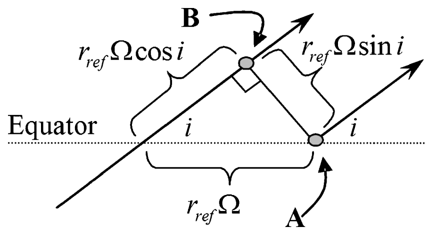

Fig. 2 Changes in orbital period due to J_2.

General Form of Cross-Track Motion

Combining Eqs. (1–6), we are left with the final form of the cross-track motion:

$$z = r_{\text{ref}}\Phi\sin(knt - \gamma) \tag{7}$$

where γ and Φ are defined by Eq. (3). Equation (7), in its current state, accurately defines the cross-track motion. However, due to the complexity in γ and Φ, the motion resulting from their substitution into Eq. (7) is not easily interpreted. The goal of this paper is to rewrite the cross-track motion parameters in a differential form that is simple, intuitive, and still has an accurate analytic solution.

Deriving the Cross-Track Equations of Motion

Instead of trying to simplify the equations for γ and Φ by linearizing them with respect to time, as was done in the previous paper,[1] γ and Φ will be treated as state variables. Now, z is no longer considered a state but is instead calculated based on these two new states. The problem now is to develop analytical equations for the motion of γ and Φ that can be easily represented in differential form. This is accomplished by using small angle theorems.

Small Angle Approximation

To begin the derivation, the small angles are identified:

$$|\Delta i \equiv (i_{\text{sat}} - i_{\text{ref}})| \ll 1, \qquad |\Delta\Omega \equiv (\Omega_{\text{sat}} - \Omega_{\text{ref}})| \ll 1$$
$$|\Phi| \ll 1 \tag{8}$$

With reference to Eq. (3), substituting the small angles into the equations and simplifying results in the following equations:

$$\gamma = \cot^{-1}[\Delta i/(\Delta\Omega\sin i_{\text{ref}})], \qquad \Phi = \sqrt{\Delta i^2 + \Delta\Omega^2\sin^2 i_{\text{ref}}} \tag{9}$$

Δi and $\Delta\Omega$ can be solved in terms of γ and Φ:

$$\Delta i = \Phi\cos\gamma, \qquad \Delta\Omega = \Phi(\sin\gamma/\sin i_{\text{ref}}) \tag{10}$$

Generating the Differential Equations

The differential equations of motion for γ and Φ can be found by taking the time derivative of Eq. (9):

$$\dot{\gamma} = \sin^2\gamma\left(-\Delta\Omega\frac{\mathrm{d}\Delta i}{\mathrm{d}t} + \Delta i\frac{\mathrm{d}\Delta\Omega}{\mathrm{d}t}\right)\bigg/(\sin i_{\text{ref}})\Delta\Omega^2$$
$$\dot{\Phi} = \left[\Delta i\frac{\mathrm{d}\Delta i}{\mathrm{d}t} + \left(\sin^2 i_{\text{ref}}\right)\Delta\Omega\frac{\mathrm{d}\Delta\Omega}{\mathrm{d}t}\right]\bigg/\Phi \tag{11}$$

Effects of the J_2 Geopotential

By further examination of the mean variations of the orbital elements and by incorporation of the small angles, the effect of J_2 on Δi and $\Delta\Omega$ is

$$\frac{\mathrm{d}\Delta\Omega}{\mathrm{d}t} = jn\Delta i\sin i_{\text{ref}}, \qquad \frac{\mathrm{d}\Delta i}{\mathrm{d}t} = 0 \tag{12}$$

By substitution of Eq. (12) into Eq. (11),

$$\dot{\Phi} = nb\frac{\Delta i\,\Delta\Omega\sin i_{\text{ref}}}{\Phi}, \qquad \dot{\gamma} = nb\frac{\Delta i^2\sin^2\gamma}{\Delta\Omega^2\sin^2 i_{\text{ref}}}$$
$$b = j\sin^2 i_{\text{ref}} \tag{13}$$

Substituting Eq. (10) into Eq. (13) results in the final form of the cross-track differential equations,

$$\dot{\Phi} = nb\Phi\cos\gamma\sin\gamma, \qquad \dot{\gamma} = nb\cos^2\gamma \tag{14}$$

There are a few very important things to note. First, the equations are nonlinear; however, they can be solved analytically. Second, the constants n and b do not depend on the current or initial state of the satellite; rather, they are only dependent on the reference orbit. This is a great improvement over the previous cross-track solution in which the forcing function was dependent on the initial conditions of the satellite.

Cross-Track Solutions

The equation for Φ is dependent on γ; however, γ is not dependent on Φ and is, therefore, solved first:

$$\gamma = \tan^{-1}(nbt + \tan\gamma_0) \tag{15}$$

Substituting the solution for γ into the differential equation for Φ and solving yields

$$\Phi = \Phi_0(\cos\gamma_0)\sqrt{\{1 + [nbt + \tan(\gamma_0)]\}^2}$$
$$\Phi = \Phi_0(\cos\gamma_0)\sec\gamma \tag{16}$$

Alternate Form of the State Equations

Equation (10) also gives rise to another set of equations for the cross-track motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tan\gamma) = nb, \qquad \Phi(\cos\gamma) = \Phi_0(\cos\gamma_0) = \Delta i \tag{17}$$

where nb and Δi are constants.

Describing the Cross-Track Motion

The cross-track motion can be described by two effects: the scissoring effect and formation separation. The scissoring effect refers to the movement of the location of intersection between two orbital planes as defined by γ. The previous equations,[1] due to their linearizations, modeled the movement of the intersection with a constant velocity, but the new equations do not suffer from this limitation. From Eq. (15), we can see that the intersection will move quickly away from the equator and asymptotically approach the poles. This effect has been called the scissoring effect because it resembles the intersection of two scissor blades as they are opened. The intersection moves quickly away from the tips, but asymptotically approaches the pivot as the scissors are opened.

Formation expansion describes the increasing amplitude of the cross-track motion. In the previous paper, this effect was captured by a linear forcing function that caused a linear change in Φ. The actual motion resembles a hyperbolic function more than a straight line. (A quadratic forcing function has been used with the equations from the previous paper, generating improved results.) Unlike the previous equations, Eq. (16) captures the variation in this motion.

Relative Motion

In the previous work, different in-plane equations of motion were derived that describe both the relative motion between a satellite and the reference orbit (x, y, z) and the relative motion between two satellites $(\Delta x, \Delta y, \Delta z)$. The equations for the cross-track motion are the same in both cases. The constants (b, c, g, j, k, n, s) are also the same in both cases because they are based solely on the reference orbit. The only difference is that the initial conditions are taken with respect to either the reference orbit or the second satellite.

Final Differential Equations and Solutions

When Eq. (14) is incorporated with the original in-plane differential equations of motion,[1] the final differential equations for the relative motion between two satellites are

$$\Delta\ddot{x} - 2nc\Delta\dot{y} - (5c^2 - 2)n^2\Delta x = 0, \qquad \Delta\ddot{y} + 2nc\Delta\dot{x} = 0$$
$$\dot{\gamma} - nb\cos\gamma = 0, \qquad \dot{\Phi} - nb\Phi\cos\gamma\sin\gamma = 0 \tag{18}$$

where $\Delta z = r_{\text{ref}}\Phi\sin(knt - \gamma)$.

We now have four differential equations of motion. Their solutions, the derived zero-drift initial conditions, and associated nondimensional constants appear hereafter for completeness.

The solutions to the differential equations of motion are

$$\Delta x = \Delta x_0 \cos(gnt) + (g/2c)\Delta y_0 \sin(gnt)$$

$$\Delta y = -(2c/g)\Delta x_0 \sin(gnt) + \Delta y_0 \cos(gnt)$$

$$\Delta z = r_{\text{ref}}\Phi \sin(knt - \gamma), \qquad \Phi = \Phi_0(\cos\gamma_0)(\sec\gamma)$$

$$\gamma = \tan^{-1}(bnt + \tan\gamma_0) \tag{19}$$

The initial conditions are

$$\Delta\dot{x}_0 = (n\Delta y_0/2)(g^2/c), \qquad \Delta\dot{y}_0 = -2cn\Delta x_0$$

$$\Delta z_0 = r_{\text{ref}}\Phi_0 \sin(-\gamma_0), \qquad \Delta\dot{z}_0 = (k-b)nr_{\text{ref}}\Phi_0\cos(\gamma_0)$$

$$\Phi_0 = (1/r_{\text{ref}})\sqrt{\Delta z_0^2 + [\Delta\dot{z}_0/(k-b)n]^2}$$

$$\gamma_0 = -a\tan^{-2}\{[\Delta\dot{z}_0/n(k-b)], \Delta z_0\} \tag{20}$$

and the constants are

$$j = \left(3J_2R_e^2\right)\big/2r_{\text{ref}}^2, \qquad n = \sqrt{\mu\big/r_{\text{ref}}^3}$$

$$s = j[(1 + 3\cos 2i_{\text{ref}})/4], \qquad b = j\sin^2 i_{\text{ref}}$$

$$k = c + j\cos^2 i_{\text{ref}}, \qquad c = \sqrt{1+s}, \qquad g = \sqrt{1-s} \tag{21}$$

Conclusions

This Note completes the work begun in the previous paper[1] by using small angles to simplify the equations of motion in the cross-track direction. The final set of equations accurately and simply describe the relative motion of satellites flying in formation in the presence of the Earth's J_2 gravitational geopotential.

Reference

[1]Schweighart, S. A., and Sedwick, R. J., "High-Fidelity Linearized J_2 Model for Satellite Formation Flight," *Journal of Guidance, Control, and Dynamics,* Vol. 25, No. 6, 2002, pp. 1073–1080.